What do Plant Stems and Repulsive Particles on a Cylinder Have in Common?

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Introduction

• Boundaries and symmetry in confined 2D particulate systems affect the structure and formation of ground states. Recent studies investigate a variety of systems including cylinders [4], parabolic confinement [5], and hard disks/spheres [2, 3].

• We study a system with a periodic boundary condition that generates a set of phyllotactic states. By combining analytic and numerical results, we obtain the ground state phase diagram. Our results have striking similarity to the formation of leaf bases on stems; the spiralling patterns they form are described by the same mathematics that describes the ground states we find in our system.

Calculating States

• Assuming a uniform density, ρ, on an infinitely long cylinder of circumference, c, and a periodic boundary condition, two lattice vectors, $\mathbf{a}$ and $\mathbf{b}$, are constrained by $m\mathbf{a} + n\mathbf{b} = \hat{\mathbf{y}}c$ and $\hat{\mathbf{z}} \cdot (\mathbf{b} \times \mathbf{a}) = c/\rho$, for integer values of m and n.

• We set $c = \rho = \sqrt{\alpha}$ for convenience. Solving the constraints gives a numerical 1D minimisation problem for a parameter, β, with respect to system energy.

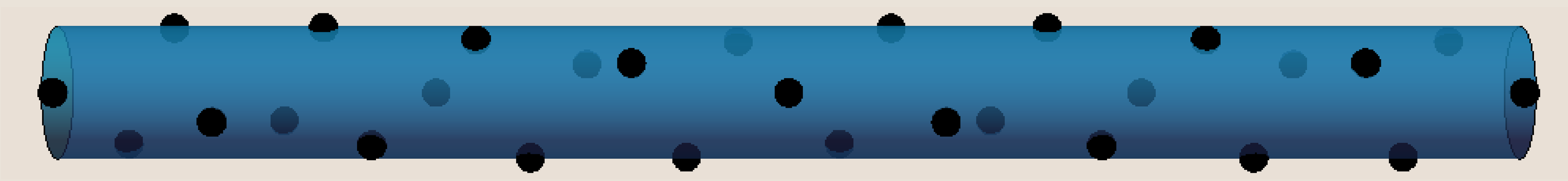
$$\mathbf{a} = \begin{pmatrix} -n/\sqrt{\alpha} \\ \beta \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} m/\sqrt{\alpha} \\ (\sqrt{\alpha} - n\beta)/m \end{pmatrix}$$

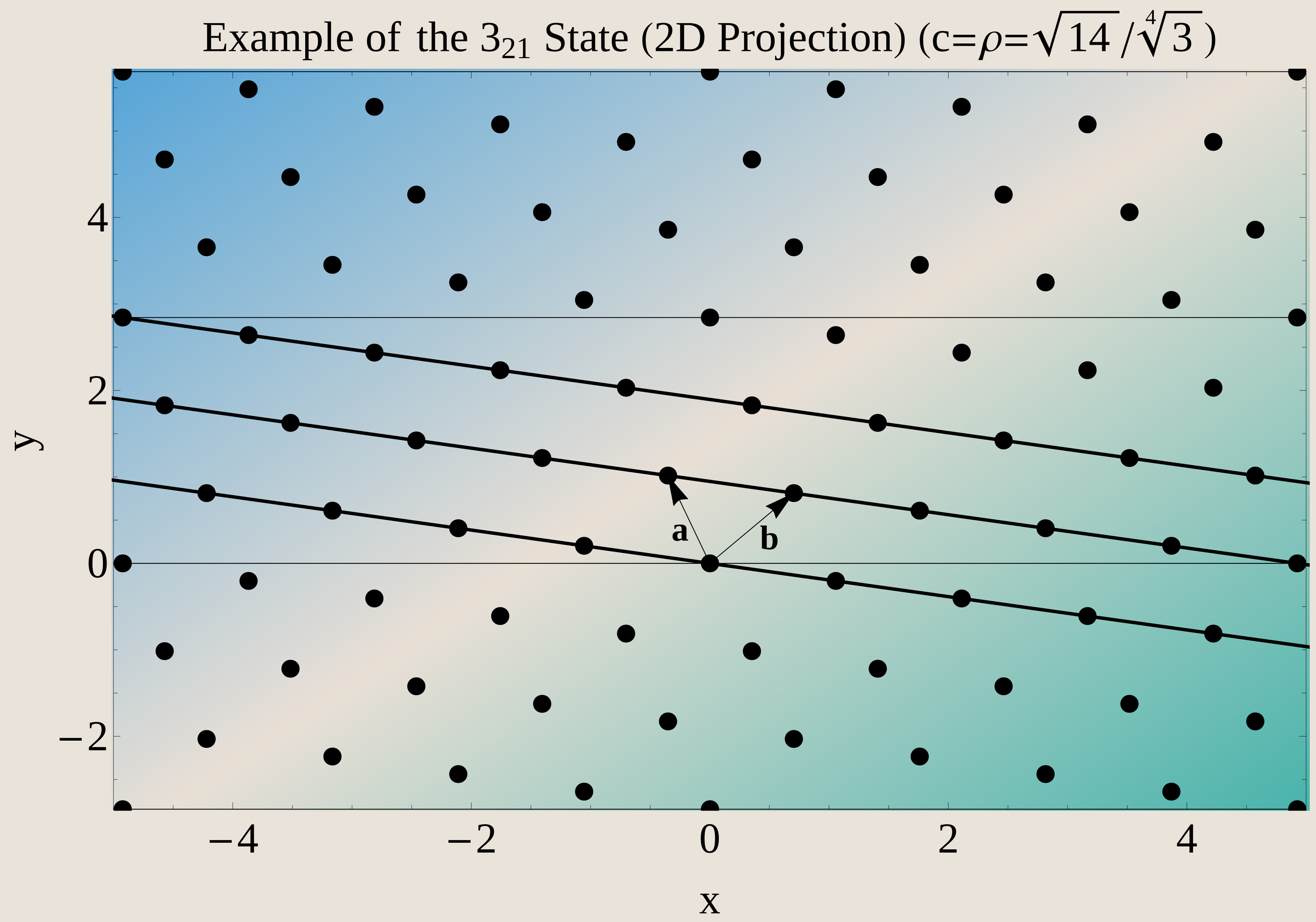
• We calculate $\beta(\alpha)$ for every $\{m, n\}$ and numerically search for ground states.

Cylindrical Geometry

• Particles on a cylinder are imaged on each revolution around the azimuthal axis. We model the system as a 2D flat space with y-periodicity representing the circumference. Below is an example of the "3_{21}" state on the cylinder and its projection in 2D. We assume the interaction follows the surface of the cylinder, with an insulating core preventing a single direct line of sight between particles.



• A choice of m and n gives a system of $N = m + n$ chains on the cylinder. We use the notation N_{mn} to describe this. Note: The unit vectors we find might not be primitive, so phyllotactic notation can be used in addition to our notation for clarity [1]. The Epilobium Ciliatum below shows a (2,1,1) (or 2_{11}) state of leaves forming on the stem.



Analytic and Numerical Results

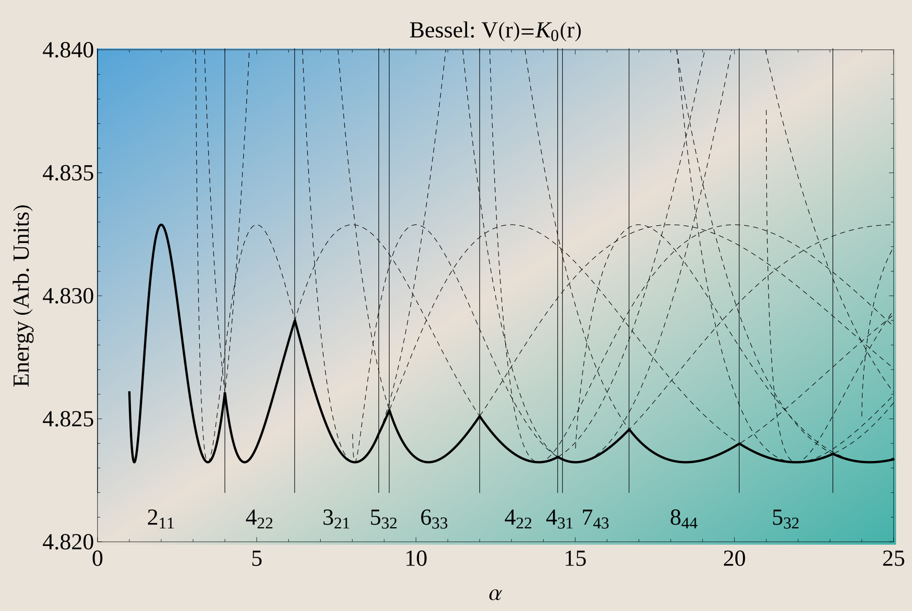
• Numerical minimisation reveals the lattice vectors are equal in length.

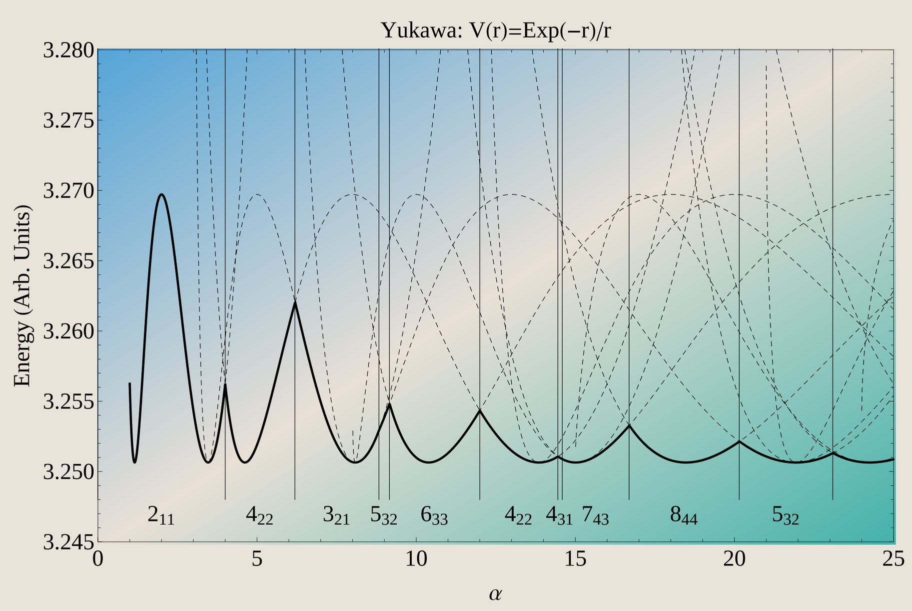
$$\beta = \frac{m\alpha - n\sqrt{\alpha^2 - (m^2 - n^2)^2}}{(m^2 - n^2)\sqrt{\alpha}}$$

• States find an absolute minimum in a perfect hexatic lattice where either $|\mathbf{a} - \mathbf{b}|$ or $|\mathbf{a} + \mathbf{b}|$ are equal to the lattice vector lengths. We find special values of α_H:

$$\alpha_H = \tfrac{2}{\sqrt{3}}(m^2 - mn + n^2) \qquad \alpha_H = \tfrac{2}{\sqrt{3}}(m^2 + mn + n^2)$$

• Along the line $c = \alpha/\rho$ we calculate the ground state energies and find transition points. Instantaneous degenerate points occur at minima for equivalent values of α_H. Transition points are exactly calculated when structures are geometrically identical or numerically calculated when the choice of potential causes non-linear energetic effects. Ground state energies are shown below for two potentials. Dashed lines show competing energies from each state and the solid line shows the ground state energy.





• All transitions have a first order discontinuity in the ground state energy. There is a general increase in row number as α increases due to the index dependence on the two hexatic minima.

Conclusion

• Phyllotaxis in nature provides us a mathematical basis from which we calculate states in our own system.

• We find ground states and transition points by combining analytic and numerical results. Some transitions are dependent on the choice of potential and a majority are determined geometrically. For $N > 1$ chains, other studies find the same energetic discontinuities at transitions as we do, giving some generality to our results.

References

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